without reference to the determinant of failure. It has already been shown that if a structure of any number of simple supports be so spaced as to be critical under the loading, then the addition of one more bay at the end will have a strengthening effect, provided that the length of the bay is not greater than the length of Euler's strut. Remembering that the substitution of a clamped for a simple support is ultimately merely the addition of another bay of infinitely small length, it follows that clamping one or both ends is equivalent to a direct addition of strength. The exact determination of the magnitude of this increased safety is, of course, to be obtained by finding the smallest root of the determinant regarded as an equation in F. In fact, the whole problem of the most efficient disposition of the supports to provide the strongest structure, in the sense that the critical longitudinal force is a maximum among those for all possible dispositions, can be treated with comparative simplicity by a discussion of the determinants already set down.

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The Electromagnetic Inertia of the Lorentz Electron.

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1. In a recent paper* on "The Effective Inertia of Electrified Systems moving with High Speed," G. W. Walker extends his previous investigation to the case of a perfectly conducting oblate spheroid, whose axis lies in the direction of motion, for the time being, of its centre, and whose eccentricity is equal to k, where k denotes the ratio of the speed of the centre to that of light, also for the instant under consideration. He finds (Note 1) that the longitudinal and transverse masses are equal respectively to

$$2e^2(1-\tfrac{1}{5}k^2)/3ac^2(1-k^2)^{3/2} \quad \text{and} \quad 2e^2(1+\tfrac{1}{60}k^2)/3ac^2(1-k^3)^{1/2}, \tag{1}$$

instead of

$$2e^2/3ac^2(1-k^2)^{3/2} \quad \text{and} \quad 2e^2/3ac^2(1-k^2)^{1/2}, \tag{2}$$

the values given by the mass formulæ of Lorentz and required by the Principle of Relativity.

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* 'Roy. Soc. Proc.,' A, vol. 93, p. 448 (1917).

The method used by Walker obviates the objections raised to the assumption of quasi-stationary motion, which is necessary in the ordinary energy method of calculating the masses, but it requires special assumptions to be made at the outset both as to the constitution of the electron and as to the boundary conditions at its surface whenever definite information is needed. These assumptions obviously limit the generality of the investigation, so that it is open to the objection that Walker's model of the electron is not a suitable one, precisely because it does not agree with the Principle of Relativity.

2. About ten years ago I developed a method of calculating the electromagnetic masses of an electron from the fundamental equations of the Electron Theory, without making any assumptions respecting either the structure of the electron or the nature of its motion, whether quasi-stationary or otherwise, until the conclusion of the general part of the investigation. This formed the basis of Chapter XI of my Adams Prize Essay, and was published in full in my book on 'Electromagnetic Radiation,' Appendices C and D. (Note 2.)

In this method the usual retarded potentials due to an element of charge moving in any assigned manner are developed in series, of which the first terms are the potentials of the element due to uniform motion with the velocity at the instant considered, whilst the succeeding terms are derived from them by definitely assigned operations, and involve the accelerations of various orders. The electric, magnetic, and mechanical forces on a second element of charge are derived by the usual operations, and the resultant mechanical force on the electron is obtained by a double integration over all the pairs of elements of charge in the form of a series proceeding according to ascending powers of a length a, which measures the linear dimensions of the electron—the radius for the Abraham electron, the transverse radius for the Lorentz electron. The first two terms of this series for the internal electromagnetic force **F** on the electron are of the form

$$\mathbf{F} = -\frac{d\mathbf{G}}{dt} + \mathbf{K}.^{*} \tag{3}$$

G may be identified with the electromagnetic momentum of the electron; it is of the order a^{-1}, and depends only on the configuration of the electron, on the relative motion of its parts, and on the velocity of its centre, all for the instant under consideration. It gives rise to the principal term in the expression for the resultant internal electromagnetic force, the only term found in the ordinary energy method.

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[*] *Loc. cit.*, p. 246, equation (341).

K represents the reaction due to radiation; it does not depend on the configuration of the electron, nor on the relative motion of its parts, but only on the velocity, acceleration, and acceleration of the second order of its centre. It is of order zero in a, and is very small compared with the principal term, except when the speed is nearly equal to that of light and the accelerations are very large. On p. 183 of my book I have estimated the error arising from neglecting it in the case of a β-particle moving in an electrostatic field of 1,000,000 volts per centimetre. With a minimum speed of 0·999 of that of light, I find it to be less than two parts in 10,000,000,000. For an electron moving in an orbit of atomic dimensions, the error would be much greater, but still quite unimportant, unless the speed were very nearly equal to that of light.

The next term of the series is of order a and depends on the velocity and the accelerations of the first, second, and third orders, and so on. These higher terms could be calculated if it were worth while, but in all practical cases we can obtain a sufficient idea of the rate of convergence of the series from its first two terms. It is important to notice that, according to my results, the error made in assuming quasi-stationary motion arises from neglecting **K** and the higher terms, but cannot affect the principal term.

3. By applying this method to the Lorentz electron* I obtain precisely the two Lorentz mass formulæ, in full agreement with the ordinary energy method, but differing from Walker's result. The discrepancy is all the more serious in that it occurs in the principal term, and for this reason casts grave doubts on the accuracy of the most fundamental investigations of the Electron Theory.

In looking for the cause of this discrepancy one is struck at once by the fact that Walker assumes k (his symbol for the ratio of the speed of the electron to that of light, usually denoted by β) to be constant. On p. 452 of his paper he writes: "It might be argued that I ought to have supposed that the contracted electron should alter in shape as the acceleration proceeds. But the difficulty is to know how it alters. There is no correspondence between accelerated motions at different speeds such as can be proved for uniform speeds, for the transformation entirely breaks down if the velocity is not constant. One might, as a pure hypothesis which has no *a priori* justification, assume that the surface is deformed according to the same law as holds for uniform velocity. This could be worked out, and might lead to a first order term (in acceleration) for longitudinal inertia, but clearly would lead to only second order terms for transverse inertia. But this appears to me too speculative to be of any value at present."

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* *Loc. cit.*, p. 257.

4. It appears to me that it is of the very essence of the Lorentz electron that its shape should alter as its acceleration proceeds, and that in such a manner as to correspond to its speed at each instant; at any rate, that is the assumption tacitly made in my own investigation and, I believe, in all others except Walker's. At the same time, I have thought it worth while to follow out the consequences of Walker's assumption of a constant k by my own method, as I had the material for such an investigation ready to hand. On p. 260 of 'Electromagnetic Radiation' I have given expressions for the electromagnetic masses of a deformed Lorentz electron, which can be utilised for this purpose. The deformation there contemplated is supposed to be small and constant, whilst that of Walker's electron increases in proportion to the time t measured from the instant at which the speed is kc. But t is to be made zero ultimately, so that the smallness creates no difficulty. The electromagnetic momentum $\mathbf{G}$, on the other hand, depends on the relative velocities of the parts of the electron, so that the variability of the deformation of Walker's electron from Lorentz's form might be expected to cause difficulty. This, however, is not so, for I have shown on p. 247 of my book that the relative motion term in the electromagnetic momentum vanishes identically for an electron which, like Lorentz's electron, is symmetrical with respect to the centre.

5. In using the expressions in question we must bear in mind that the actual electron at time t, which is approximately of Lorentz's form, is for convenience of calculation transformed by stretching in the direction of motion in the ratio $\sqrt{(1-\beta^2)}:1$, where β denotes the ratio of the speed of the electron at time t to that of light, a process which changes the Lorentz electron with axes $a\{\sqrt{(1-\beta^2)}, 1, 1\}$ into a sphere of radius a. The electron thus transformed is assumed to be given by the equation of its surface

$$r = a + \epsilon \mathrm{P}_2(\mu). \tag{4}$$

The quantity ϵ is so small that its square and higher powers can be neglected; the axis of the spherical harmonic is supposed to make an angle ψ with the direction of motion of the electron measured in the osculating plane of its orbit towards the principal normal. The co-ordinates ξ, η, ζ, of an element of the transformed electron are referred to the tangent, principal normal, and bi-normal of the orbit as axes. The tangential and normal components of the electromagnetic momentum are given by the equations

$$\mathrm{G}_\xi = \frac{2e^2v}{3c^2a\sqrt{(1-\beta^2)}}\left\{1+\frac{\epsilon}{10a}\left(3\cos^2\psi-1\right)\right\}, \tag{5}$$

$$\mathrm{G}_\eta = \frac{e^2v\epsilon}{5c^2a^2}\sin\psi\cos\psi. \tag{6}$$

Here $v = c\beta$ and denotes the speed of the electron at the time t, and the expressions apply to a surface charge, of uniform density on the transformed Lorentz electron, but of density proportional to the perpendicular from the centre on the tangent plane of the actual Lorentz electron before deformation.

6. We shall first apply these expressions to Walker's first case of an electron moving in a straight line with acceleration μ (in his notation). At time t the speed is given by $\beta = k+\mu t/c$, while ψ is zero. From (4) we find that the tangential and transverse radii of the transformed electron are equal to $a+\epsilon$ and $a-\frac{1}{2}\epsilon$ respectively; hence those of the actual electron are equal to $(a+\epsilon)\sqrt{(1-\beta^2)}$ and $a-\frac{1}{2}\epsilon$ respectively. But the same values for Walker's electron are $a_0\sqrt{(1-k^2)}$ and a_0 respectively, where the suffix is used for the moment to distinguish the two electrons. Equating corresponding values we find to the first order in μ

$$a\sqrt{(1-\beta^2)} = a_0\sqrt{(1-k^2)}\{1-2k\mu t/3(1-k^2)c\}, \quad \epsilon/a = 2k\mu t/3(1-k^2)c.$$

Substituting these values in (5) we find, putting $\psi = 0$, and a for a_0,

$$G_\xi = \frac{2e^2}{3c^2a\sqrt{(1-k^2)}}\left\{ck + \frac{1-\frac{1}{5}k^2}{1-k^2}\mu t\right\},$$

while G_η obviously vanishes. Using (3) and differentiating on the supposition that k is constant we obtain for the resultant internal electromagnetic force on the electron

$$F_\xi = -\frac{dG_\xi}{dt} = -\frac{2e^2(1-\frac{1}{5}k^2)}{3c^2a(1-k^2)^{3/2}}\mu, \tag{7}$$

whilst F_η vanishes.

The coefficient of the acceleration μ in (7) is the longitudinal electromagnetic mass and agrees with the first of Walker's expressions (1).

7. Walker's second case is more difficult to investigate. So far as I understand his procedure from the very brief description given in his paper, he seems to treat the electron as a rigid perfect conductor, which coincides at the instant $t = 0$ with the Lorentz electron as before, but moves without rotation, so that at the time t its axis deviates from that of the Lorentz electron by the small angle ωt, measured in the osculating plane towards the outside of the orbit. Here ω denotes the angular velocity of the electron in its orbit, so that the acceleration μ is equal to ωv, where v is constantly equal to ck. Thus the equation of the surface of Walker's electron referred to the tangent, principal normal, and binormal of the orbit at time t as axes becomes on this supposition

$$\frac{(x-y\omega t)^2}{1-k^2}+(y+x\omega t)^2+z^2 = a^2 \quad \text{or} \quad \frac{x^2}{1-k^2}+y^2+z^2-\frac{2k^2\omega t}{1-k^2}xy = a^2$$

to the first order in the small angle ωt. Applying the inverse Lorentz transformation $x = \xi\sqrt{(1-k^2)}$, $y = \eta$, $z = \zeta$, we find that the equation of the transformed Walker's electron is

$$\xi^2+\eta^2+\zeta^2-\frac{2k^2\omega t}{\sqrt{(1-k^2)}}\,\xi\eta = a^2.$$

Two principal axes lie in the osculating plane and make angles $\psi_1 = 45^\circ$ and $\psi_2 = 135^\circ$ with the tangent, whilst the lengths of the corresponding semi-axes are equal to $a\{1 \pm k^2\omega t/2\sqrt{(1-k^2)}\}$ respectively.

On the other hand we may write the equation in the form

$$r = a+\epsilon_1 P_2(\mu_1)+\epsilon_2 P_2(\mu_2) \qquad (8)$$

analogous to (4), where μ_1 is measured from the axis ψ_1 and μ_2 from ψ_2. Putting $\mu_1 = 1$, $\mu_2 = 0$, and $\mu_1 = 0$, $\mu_2 = 1$, alternately we find that the two principal semi-axes are $a+\epsilon_1-\frac{1}{2}\epsilon_2$ and $a-\frac{1}{2}\epsilon_1+\epsilon_2$ respectively. Comparing the two pairs of values we obtain

$$\epsilon_1 = -\epsilon_2 = ak^2\omega t/3\sqrt{(1-k^2)}. \qquad (9)$$

Owing to the smallness of ϵ_1 and ϵ_2, the principle of superposition holds and (5) and (6) each contain two terms corresponding to ϵ_1 and ϵ_2. In the first the two terms clearly cancel each other, in the second they are equal. Bearing in mind that, in the present case, β and k are identical, we find

$$G_\xi = \frac{2e^2v}{3c^2a\sqrt{(1-k^2)}}, \qquad G_\eta = \frac{e^2k^2v\omega t}{15c^2a\sqrt{(1-k^2)}}.$$

In using (3) we must bear in mind that the axes are rotating with the angular velocity ω, hence we find that the resultant internal electromagnetic force on the electron at the time $t = 0$ is given by

$$F_\xi = -\frac{dG_\xi}{dt}+\omega G_\eta = 0, \qquad F_\eta = -\frac{dG_\eta}{dt}-\omega G_\xi = -\frac{2e^2(1+\frac{1}{10}k^2)\,v\omega}{3c^2a\sqrt{(1-k^2)}}. \qquad (10)$$

The value of F_η differs from that found by Walker by having a term $\frac{1}{10}k^2$ inside the bracket in the numerator in place of $\frac{1}{60}k^2$.

8. One difficulty remains which requires examination. In the three preceding sections we have assumed the surface density of the transformed electron to be uniform, or, what amounts to the same thing, that of the actual electron to be proportional to p, the perpendicular from the centre on the tangent plane. But the surface density of Walker's electron is not proportional to p simply; in the first case it involves a small term, of the first order in the acceleration, proportional to px, and, in the second case, similar small terms proportional to py, as we see from his expressions for σ given on p. 450 and on p. 451 respectively. I shall now prove that these small additional terms in the surface density do not affect the expressions

(7) and (10) so long as we confine our investigation to terms of the first order. The proof depends upon the following propositions, which are stated as briefly as possible:—

(1) Suppose that a slightly deformed Lorentz electron of uniform volume density is transformed by the Lorentz transformation $x = \xi\sqrt{(1-\beta^2)}$, $y = \eta$, $z = \zeta$, so that the equation of the bounding surface of the transformed electron becomes of the form $r = a+\Sigma\epsilon \mathrm{P}_i(\mu)$.

Then the only terms which influence the electromagnetic masses are those for which $i = 2$.

This proposition is proved on p. 260 of 'Electromagnetic Radiation.'

(2) Let the equation of the transformed surface be of the form

$$f(\xi, \eta, \zeta) = r-a-\Sigma\epsilon \mathrm{Z}_i = 0, \tag{11}$$

where Z_i is a homogeneous function of ξ, η, ζ of degree i, for instance, the solid harmonic associated with P_i, and ϵ is a small coefficient as before.

Then the perpendicular from the origin on the tangent plane ϖ is equal to the radius r, squares and products of the ϵ's being neglected.

In fact, if the direction cosines of ϖ be λ, μ, ν, we have in succession,

$$\frac{\partial f}{\partial \xi} = \frac{\xi}{r} - \Sigma\epsilon\frac{\partial \mathrm{Z}_i}{\partial \xi}, \text{ etc.,} \qquad \sqrt{\left\{\left(\frac{\partial f}{\partial \xi}\right)^2 + \left(\frac{\partial f}{\partial \eta}\right)^2 + \left(\frac{\partial f}{\partial \zeta}\right)^2\right\}} = 1 - \Sigma\epsilon\frac{i\mathrm{Z}_i}{r},$$

$$\lambda = \frac{\partial f}{\partial \xi}\Big/\sqrt{\left\{\left(\frac{\partial f}{\partial \xi}\right)^2 + \left(\frac{\partial f}{\partial \eta}\right)^2 + \left(\frac{\partial f}{\partial \zeta}\right)^2\right\}} = \left\{\frac{\xi}{r} - \Sigma\epsilon\frac{\partial \mathrm{Z}_i}{\partial \xi}\right\}\left\{1 + \Sigma\epsilon\frac{i\mathrm{Z}_i}{r}\right\}, \text{ etc.,}$$

$$\varpi = \lambda\xi + \mu\eta + \nu\zeta = \{r - \Sigma\epsilon i\mathrm{Z}_i\}\left\{1 + \Sigma\epsilon\frac{i\mathrm{Z}_i}{r}\right\} = r + \text{terms of order } \epsilon^2.$$

(3) Consider a homœoid of uniform volume density A, bounded by the surfaces with radii r' and $(1+\alpha)r'$, where

$$r' = r\{1+\gamma \mathrm{S}_n\} = a+\Sigma\epsilon \mathrm{Z}_i + a\gamma \mathrm{S}_n. \tag{12}$$

A is infinitely large and α infinitely small, but so that $\mathrm{A}\alpha$ is finite and equal to $e/4\pi a^3$ ultimately. S_n is a homogeneous function of ξ, η, ζ of degree n, for instance, a spherical harmonic of order n, and γ is a small coefficient of the order of ϵ.

Then the homœoid is equivalent to a surface distribution of density τ on the surface (11), where

$$\tau = er\{1+\gamma \mathrm{S}_n\}/4\pi a^3 = e\varpi\{1+\gamma \mathrm{S}_n\}/4\pi a^3. \tag{13}$$

In fact, the thickness of the homœoid is equal to $\alpha r'$, and the surface density to the limit of $\mathrm{A}\alpha r'$, whilst $r = \varpi$ to the first order by proposition (2).

(4) Let us regard (11) as a transformed electron, to which corresponds an

actual electron with surface density σ, and perpendicular on the tangent plane p. Then the surface densities σ and τ are connected by the equation

$$\sigma = \tau p / \varpi \sqrt{(1-\beta^2)}. \tag{14}$$

Apply the Lorentz transformation, bearing in mind that it leaves elements of charge unaltered, but changes elements of volume in the ratio $\sqrt{(1-\beta^2)} : 1$. If corresponding surface elements be dS and $d\Sigma$, then corresponding elements of charge are σdS and $\tau d\Sigma$, while corresponding elements of volume are $p dS$ and $\varpi d\Sigma$. Hence we have

$$\sigma dS = \tau d\Sigma, \qquad p dS = \varpi d\Sigma \sqrt{(1-\beta^2)},$$

whence the result follows.

(5) Replacing Z_i and S_n in (11), (12), and (13) by $r^i P_i$ and $r^n P_n$ respectively, where the P's denote zonal harmonics as usual, we may write a in place of r in the small terms; hence it follows that the total charge of the homœoid in (3) is equal to e.

Then we may express the result of the last section as follows:—

Let the uniform homœoid, with total charge e, whose inner surface is given by (12), where Z_i and S_n now denote spherical harmonics, be transformed by the Lorentz transformation appropriate to the motion for the time being of its centre, viz., $\xi = x/\sqrt{(1-\beta^2)}$, $\eta = y$, $\zeta = z$ Then the transformed homœoid is equivalent to a charge of density σ distributed over the surface whose equation is (11), where

$$\sigma = ep\{1+\gamma S\}/4\pi a^3 \sqrt{(1-\beta^2)}. \tag{15}$$

9. In order to apply this result to Walker's electrons, for which σ includes terms proportional to x and y respectively, we need only take Z_i proportional to the zonal harmonics P_i used in (4) and (8) respectively, and S_n proportional to x and y respectively, so that n is unity in each case.

Then the corresponding transformed electrons are equivalent to uniform homœoids, whose inner boundaries are given by equations of the type (12) with $n = 1$, and $i = 2$ as before.

By proposition (1) the first small deviation from the Lorentz electron does not influence the electromagnetic masses, whilst the second gives the expressions (7) and (10) respectively as before.

10. It has been already stated in Section 4 that the electromagnetic momentum **G** depends on the relative motion of the parts of the electron, as well as on its configuration, and it might be thought that the relative motion term might account for the discrepancy between (10) and Walker's expression for the transverse mass, but this is not so.

The relative motion term in question is given by (343) on p. 247 of 'Electromagnetic Radiation'; it is of order zero in a, but even if it were of order -1, it would disappear. For in the Lorentz electron there is rotation, but there is also symmetry with respect to the centre; on the other hand, in Walker's electron there is no symmetry with respect to the centre; but, if I understand Walker aright, there is no rotation and therefore no relative motion.

A clue is afforded by the presence of the last term, $+\frac{2}{3}Bk^2(1-k^2)^{-1}vpa^{-3}$, in the expression for $4\pi\sigma$ on p. 451 of Walker's paper, for this term when multiplied by the principal term in Q and integrated over the electron accounts for the discrepancy; but I have not been able to trace it to its source from the very brief indications given. (Notes 1 and 3.)

Be that as it may, the general agreement (apart from this discrepancy) between the results obtained by such widely different methods must be regarded as a welcome confirmation of the accuracy of the calculations, and in my opinion leaves little doubt that the differences between Walker's and Lorentz's electromagnetic masses are after all due to Walker's assumption of a contracted electron, which does not alter in shape as the acceleration proceeds (Note 4). Hence his results do not appear to me to constitute an argument against the Lorentz mass formulæ, or the Principle of Relativity, at any rate so long as his resulting mass formulæ do not afford an agreement with experiment so decidedly superior as to outweigh theoretical considerations.

I think that it will be pretty generally admitted that the mass formulæ of Lorentz have the advantage of all others from the point of view of pure theory, in so far as they are consistent with the Principle of Relativity and in addition admit of the possibility of a mechanical explanation of the electron, as shown in 'Electromagnetic Radiation,' Appendices D and E, whatever this possibility may be worth in the future. But, what is perhaps more important from the practical point of view, they afford the basis for a comparatively simple and workable mechanics of the electron, as I have shown in 'Electromagnetic Radiation,' Appendices F and G. This advantage of simplicity, which is almost essential from the point of view of economy of thought, is not shared by the more complicated formulæ, such as those of Abraham and Walker. Thus the superiority of any one of the latter over the formula of Lorentz in respect of agreement with experiment would require to be extremely marked before it could be adopted in preference.

11. When the experimental evidence is examined no marked difference is observed between the two formulæ for the transverse mass, and indeed all that Walker claims is that his formula is as good as that of Lorentz in this respect, and that the experiments so far performed cannot distinguish between them. Walker does not appear to be acquainted with the very careful and

extensive series of experiments of Neumann.* These were made by Bucherer's method and confirmed his results fully. In all 55 plates were obtained, including all preliminary attempts, of which 26 were good enough to be measured. Of these 22, taken between $\beta = 0{\cdot}4$ and $\beta = 0{\cdot}7$, gave an agreement with the Lorentz formula with a probable error in the specific charge, e/m_0, of 0·15 per cent.; the remaining 4, taken between $\beta = 0{\cdot}7$ and $\beta = 0{\cdot}8$, indicated that the specific charge, e/m_0, increased faster than it should according to the Lorentz formula, by perhaps 2 per cent. The last result is regarded as doubtful by Neumann on account of the small number of experiments and the large experimental errors to which measurements at such high speeds are liable; he proposes to extend his experiments in this direction. If the result should be confirmed it would hardly be likely to make much difference in the relative performances of the two formulæ, although it might require the addition of some proportion of nonelectromagnetic mass to the formula of Lorentz. This is a possibility that I also have had in mind for some years in connection with the effect due to the second order terms in the internal electromagnetic force, those included in the radiation pressure $\mathbf{K}$; this investigation, however, is not yet completed. Nevertheless, the results obtained so far indicate that further experiments on the specific mass of the electron for speeds from $\beta = 0{\cdot}7$ upwards are very urgently needed. For my own part I doubt whether experiments at these high speeds on the old lines can be expected to yield very good results, owing to the smallness of the deflections to be measured and the comparatively great width and diffuse character of the trace obtained on the photographic plate. In 'Electromagnetic Radiation,' p. 300, a method is given which would appear to have some advantages, but it could only be performed on a scale beyond the resources of most laboratories and possibly only in institutions possessing an equipment such as that of the National Physical Laboratory.

Notes added April 24, 1918.

(1) On repeating Walker's calculations for the transverse acceleration, I find that the sign of the discrepant term in the expressions for (X, Y, Z) and $4\pi\sigma$ on p. 451 of his paper should be positive. In consequence, the factor $1+\frac{11}{60}k^2$ in the expression for the transverse inertia on p. 452 must be replaced by $1+\frac{1}{60}k^2$.

(2) My expression for the internal electromagnetic force on the electron is based on three assumptions: (*a*) the retarded potentials of a point charge,

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* 'Ann. d. Phys.,' Folge 4, Band 45, S. 529 (1914).

according to Liénard and Wiechert; (*b*) the usual expression for the mechanical force on a moving electric charge, together with the usual rule of composition of forces; and (*c*) the assumed convergence of my expansions for the potentials and electric, magnetic, and mechanical forces. Few will cavil at (*a*) and (*b*), but (*c*) has not been, and I doubt whether it ever will be, proved for the most general type of motion. In certain cases the convergence of the series can be demonstrated, as I have proved elsewhere, for a uniform longitudinal acceleration;* they converge almost certainly in all cases where the acceleration is not too great, and the speed of the electron not too close to that of light. With this proviso, my expression for the electromagnetic momentum can be applied to any electron, whatever its constitution, form, state of strain or internal motion, and motion as a whole may be, provided that the relative velocities of its parts are small enough.

(3) I have now succeeded in tracing the causes of the discrepancy referred to in the text by applying Walker's method, but on the basis of the principles used by Lorentz and myself. Consistently with his assumption of a perfectly conducting electron and consequent zero force inside it, Walker calculates the total mechanical force on it by means of his expressions $\frac{1}{2}\int\sigma \mathrm{P}d\mathrm{S}$ and $\frac{1}{2}\int\sigma \mathrm{Q}d\mathrm{S}$ in the two cases respectively. Lorentz and myself, on the other hand, postulate the truth of the electromagnetic equations inside the charge as well as outside, and consequently admit the possibility of force inside the electron, so that on our assumptions the correct expressions for the total force on it are $\frac{1}{2}\int\sigma(\mathrm{P}+\mathrm{P}_i)d\mathrm{S}$ and $\frac{1}{2}\int\sigma(\mathrm{Q}+\mathrm{Q}_i)d\mathrm{S}$ respectively, where P_i and Q_i denote the forces just inside. For the longitudinal acceleration of Walker's rigid electron P_i vanishes, but for the transverse acceleration Q_i differs from zero.

In order to obtain a convenient formula for calculating the inside force we proceed as follows. With the usual notation the electromagnetic equations written in vector form are

$$\mathrm{C}\,\mathrm{curl}\,\mathbf{h} = \frac{\partial \mathbf{d}}{\partial t} + 4\pi\rho\mathbf{v} = \frac{\mathrm{D}\mathbf{d}}{\mathrm{D}t} - (\mathbf{v}\nabla)\mathbf{d} + 4\pi\rho\mathbf{v}, \qquad \mathrm{div}\,\mathbf{h} = 0,$$

$$\mathrm{C}\,\mathrm{curl}\,\mathbf{d} = -\frac{\partial \mathbf{h}}{\partial t} = -\frac{\mathrm{D}\mathbf{h}}{\mathrm{D}t} + (\mathbf{v}\nabla)\mathbf{h}, \qquad \mathrm{div}\,\mathbf{d} = 4\pi\rho,$$

where $\mathrm{D}/\mathrm{D}t$ denotes differentiation following the motion of the element of charge, whose volume density is ρ and velocity $\mathbf{v}$. Apply these equations to an infinitesimal element of the infinitely thin surface layer of surface density σ, which constitutes the moving electron. Since the surface discontinuity travels with the element, the differential coefficients $\mathrm{D}\mathbf{d}/\mathrm{D}t$ and

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* 'Ann. d. Phys.,' vol. 25, p. 77, § 9 (1908).

$\mathbf{Dh}/\mathrm{D}t$ remain finite, whilst the others are generally infinite Integrating throughout the volume of the surface element, we obtain in the usual way

$$\mathrm{C}\{[\mathbf{Nh}]\} = 4\pi\sigma\mathbf{v}-(\mathbf{Nv})\{\mathbf{d}\}, \qquad \{(\mathbf{Nh})\} = 0,$$
$$\mathrm{C}\{[\mathbf{Nd}]\} = (\mathbf{Nv})\{\mathbf{h}\}, \qquad \{(\mathbf{Nd})\} = 4\pi\sigma,$$

where $\mathbf{N}$ denotes a unit normal vector drawn from the inside to the outside of the surface, and brackets { } indicate that the excess of the value outside above that inside the surface is to be taken for the quantity enclosed in the brackets. Solving these equations we find

$$(\mathrm{C}^2-(\mathbf{Nv})^2)\{\mathbf{d}\} = 4\pi\sigma(\mathrm{C}^2\mathbf{N}-(\mathbf{Nv})\mathbf{v}), \qquad (\mathrm{C}^2-(\mathbf{Nv})^2)\{\mathbf{h}\} = -4\pi\sigma\mathrm{C}[\mathbf{Nv}]. \tag{16}$$

Hence we find for the mechanical force $\mathbf{f}$ on unit charge

$$(\mathrm{C}^2-(\mathbf{Nv})^2)\{\mathbf{f}\} = (\mathrm{C}^2-(\mathbf{Nv})^2)\{\mathbf{d}+[\mathbf{vh}]/\mathrm{C}\} = 4\pi\sigma(\mathrm{C}^2-\mathbf{v}^2)\mathbf{N}. \tag{17}$$

Knowing the surface density σ, the velocity $\mathbf{v}$, normal $\mathbf{N}$ and the value of $\mathbf{f}$, components (P, Q, R), just outside the surface, we can use this formula to calculate the values of $(\mathrm{P}_i, \mathrm{Q}_i, \mathrm{R}_i)$ just inside. We shall apply it to each of Walker's cases in turn, neglecting squares of μt throughout.

(a) *Walker's Longitudinal Case.*—Denoting the components of $\mathbf{N}$, the direction cosines of the outward normal at (x, y, z) by (l, m, n), we have

$$(l, m, n) = (xg^{-2}, y, z)pa^{-2}, \qquad \text{where } g^2 = 1-k^2,$$
$$\mathbf{v} = (\mathrm{C}k+\mu t, 0, 0), \qquad \mathrm{C}^2-\mathbf{v}^2 = \mathrm{C}^2g^2(1-2k\mu t\mathrm{C}^{-1}g^{-2}),$$
$$\mathrm{C}^2-(\mathbf{Nv})^2 = \mathrm{C}^2(1-k^2l^2-2l^2k\mu t\mathrm{C}^{-1}).$$

Hence we find by means of (17)

$$\begin{aligned}\{\mathrm{P}\} &= 4\pi\sigma g^2l(1-k^2l^2)^{-1}(1-2k\mu t\mathrm{C}^{-1}g^{-2}(1-l^2)(1-k^2l^2)^{-1}) \\ &= 4\pi\sigma xp^{-1}(1-2k\mu t\mathrm{C}^{-1}g^{-2}(1-x^2g^{-2}a^{-2})).\end{aligned} \tag{18}$$

This is identical with the value of P given by Walker on p. 450, when his value of σ is inserted; hence P_i is zero, and Walker's calculation of the force on the electron is correct. It is easy to verify, by (16), that $l\mathrm{X}_i+m\mathrm{Y}_i+n\mathrm{Z}_i$ vanishes at the surface, so that Walker's σ is correct.

(b) *Walker's Transverse Case.*—Here we have, using (17),

$$\mathbf{v} = (\mathrm{C}k, \mu t, 0), \quad \mathrm{C}^2-\mathbf{v}^2 = \mathrm{C}^2g^2, \quad \mathrm{C}^2-(\mathbf{Nv})^2 = \mathrm{C}^2(1-k^2l^2-2lmk\mu t\mathrm{C}^{-1}),$$
$$\begin{aligned}\{\mathrm{Q}\} &= 4\pi\sigma g^2m(1-k^2l^2)^{-1}(1+2k\mu t\mathrm{C}^{-1}lm(1-k^2l^2)^{-1}) \\ &= 4\pi\sigma g^2yp^{-1}(1+2k\mu txy\mathrm{C}^{-1}g^{-2}a^{-2}) \\ &= \mathrm{A}ya^{-3}(g^2+2k\mu txy\mathrm{C}^{-1}a^{-2}-2\mu(1+\tfrac{1}{10}k^2)y\mathrm{C}^{-2}+\tfrac{1}{3}\mu k^2y\mathrm{C}^{-2}),\end{aligned}$$

by inserting the value of $4\pi\sigma$ given by Walker on p. 451 with the sign of the discrepant last term corrected. Comparing with his value of Q we obtain

$$\mathrm{Q}_i = \mathrm{Q}-\{\mathrm{Q}\} = -\tfrac{1}{3}\mathrm{A}\mu k^2y^2\mathrm{C}^{-2}a^{-3}.$$

Thus Q_i differs from zero inside the electron owing to the presence of the discrepant term, and hence we must add to Walker's value for the force on the electron a correction, which amounts to

$$\tfrac{1}{2}\int \sigma Q_i dS = -(1/24\pi)A^2\mu k^2 C^{-2}a^{-6}\int py^2 dS = -\tfrac{1}{18}\mu e^2 k^2 C^{-2}a^{-1}g^{-1}.$$

Adding this to the corrected value of Walker's force on p. 452, we obtain

$$-\tfrac{2}{3}\mu e^2(1+\tfrac{1}{10}k^2)C^{-2}a^{-1}g^{-1},$$

which is precisely the value obtained in the text, § 7, equation (10).

Hence the discrepancy is completely accounted for.

Here also it is easy to verify that $lX_i+mY_i+nZ_i$ vanishes at the surface on account of (16), so that Walker's σ is correct.

(4) *Longitudinal Acceleration of the Lorentz Electron.*—I have now succeeded in verifying the supposition in the text for the longitudinal acceleration by adapting Walker's solution to the case of the continually deformed electron, due account being taken both of the relative motion and of the internal force.

At time t the eccentricity is $k+\mu t C^{-1} \equiv \beta$, as in § 6. Putting

$$x = \xi\sqrt{(1-\beta^2)},$$

we notice that ξ is constant for the Lorentz electron, so that there is a relative velocity $\dot{x} = -\beta\dot{\beta}x(1-\beta^2)^{-1} = -\mu kxC^{-1}g^{-2}$ to the first order. By interaction with the external magnetic force $Aka^{-3}(0, -z, y)$ at the surface this produces an additional external mechanical force $A\mu k^2C^{-2}g^{-2}a^{-3}(0, xy, xz)$. To neutralise its tangential component we may add to Walker's χ on p. 450 a term

$$\chi' = \tfrac{1}{5}A\mu k^2a^2xC^{-2}g^{-4}\rho^{-3}(1-a^2\rho^{-2}+\tfrac{5}{3}a^2x^2g^{-2}\rho^{-4}).$$

This gives rise to an additional surface density σ', given by

$$4\pi\sigma' = A\mu k^2pxC^{-2}g^{-4}a^{-3}(\tfrac{2}{5}-\tfrac{4}{3}x^2g^{-2}a^{-2}).$$

Altogether we obtain an additional external normal mechanical force (P', Q', R'), such that at the surface we have

$$P' = A\mu k^2x^2C^{-2}g^{-4}a^{-3}(\tfrac{7}{5}-\tfrac{7}{3}x^2g^{-2}a^{-2}).$$

These quantities, σ', P', are to be added to σ, P, given by Walker on p. 450, but in using the latter we must bear in mind that they refer to an electron moving with speed $Ck+\mu t$ and of the shape appropriate to this speed; hence, in the principal terms, the quantities A and p must be taken for the eccentricity β. In other words, if A and p are still to refer to eccentricity k as before, we must, in the principal terms, replace them by

$$A+\mu tC^{-1}dA/dk = A(1+k\mu tC^{-1}g^{-2}),$$

and $$p+\mu tC^{-1}dp/dk = p\{1-k\mu tC^{-1}g^{-2}(1+k^2)l^2\},$$

where $l = pxg^{-2}a^{-2}$ in the small term.

With these changes in Walker's solution we obtain

$$\mathrm{P}+\mathrm{P}' = \mathrm{A}xa^{-3}\{1-k\mu t\mathrm{C}^{-1}g^{-2}(1-2x^2g^{-2}a^{-2})$$
$$-2\mu x\mathrm{C}^{-2}g^{-4}(1-\tfrac{9}{10}k^2+\tfrac{7}{6}k^2x^2g^{-2}a^{-2})\},$$
$$4\pi(\sigma+\sigma') = \mathrm{A}pa^{-3}\{1+k\mu t\mathrm{C}^{-1}g^{-2}(1-(1+k^2)l^2)$$
$$-2\mu x\mathrm{C}^{-2}g^{-4}(1-\tfrac{2}{5}k^2+\tfrac{2}{3}k^2x^2g^{-2}a^{-2})\}.$$

In order to find $\{\mathrm{P}+\mathrm{P}'\}$, we must use (16) with

$$\mathbf{v} = (\mathrm{C}k+\mu t-\mu kx\mathrm{C}^{-1}g^{-2},\ 0,\ 0).$$

Thus, we may use (18), provided that we replace t by $t-kx\mathrm{C}^{-1}g^{-2}$, σ by $\sigma+\sigma'$, and p by the value belonging to eccentricity β. Hence we find

$$\{\mathrm{P}+\mathrm{P}'\} = \mathrm{A}xa^{-3}\{1-k\mu t\mathrm{C}^{-1}g^{-2}(1-2x^2g^{-2}a^{-2})$$
$$-2\mu x\mathrm{C}^{-2}g^{-4}(1-\tfrac{7}{5}k^2+\tfrac{5}{3}k^2x^2g^{-2}a^{-2})\}.$$

Thus we obtain

$$\mathrm{P}_i+\mathrm{P}_i' = \mathrm{P}+\mathrm{P}'-\{\mathrm{P}+\mathrm{P}'\} = -\mathrm{A}\mu k^2x^2\mathrm{C}^{-2}g^{-4}a^{-3}(1-x^2g^{-2}a^2).$$

Hence the mechanical force just inside the electron differs from zero, and the total force on it is given by

$$\tfrac{1}{2}\int(\sigma+\sigma')(\mathrm{P}+\mathrm{P}'+\mathrm{P}_i+\mathrm{P}_i')d\mathrm{S}$$
$$= -(1/2\pi)\mu\mathrm{A}^2\mathrm{C}^{-2}g^{-4}a^{-6}\int(1-\tfrac{2}{5}k^2+\tfrac{2}{3}k^2x^2g^{-2}a^{-2})x^2pd\mathrm{S} = -\tfrac{2}{3}\mu e^2\mathrm{C}^{-2}a^{-1}g^{-3},$$

all other terms being odd functions of x and disappearing by symmetry. This is precisely the expression of Lorentz.

As before, we must verify that the density is unaltered by the presence of internal force. The same process gives for $\rho = a$

$$\{\mathbf{d}+\mathbf{d}'\} =$$
$$\mathrm{A}a^{-3}\{1+k\mu t\mathrm{C}^{-1}g^{-2}(1+2x^2g^{-2}a^{-2})-2\mu x\mathrm{C}^{-2}g^{-4}(1-\tfrac{2}{5}k^2+\tfrac{5}{3}k^2x^2g^{-2}a^{-2})\}(x,\ y,\ z)$$
$$-\mathrm{A}k\mu x\mathrm{C}^{-1}g^{-2}a^{-3}(2t-2kx\mathrm{C}^{-1}g^{-2})(1,\ 0,\ 0),$$

$$\mathbf{d}+\mathbf{d}' =$$
$$\mathrm{A}a^{-3}\{1+k\mu t\mathrm{C}^{-1}g^{-2}(1+2x^2g^{-2}a^{-2})-2\mu x\mathrm{C}^{-2}g^{-4}(1-\tfrac{2}{5}k^2+\tfrac{7}{6}k^2x^2g^{-2}a^{-2})\}(x,y,z)$$
$$-\mathrm{A}k\mu x\mathrm{C}^{-1}g^{-2}a^{-3}(2t-kx\mathrm{C}^{-1}g^{-2})(1,\ 0,\ 0).$$

Hence we find

$$\mathbf{d}_\mathrm{i}+\mathbf{d}_\mathrm{i}' = \mathrm{A}\mu k^2x^3\mathrm{C}^{-2}g^{-6}a^{-5}(x,\ y,\ z)-\mathrm{A}\mu k^2x^2\mathrm{C}^{-2}g^{-4}a^{-3}(1,\ 0,\ 0).$$

Multiplying the components of $\mathbf{d}_\mathrm{i}+\mathbf{d}_\mathrm{i}'$ by $(l,\ m,\ n)$ or to the first order by $(xg^{-2},\ y,\ z)pa^{-2}$ respectively and adding, we see that the normal component of the electric force vanishes just inside the surface, so that the surface density is given correctly by the expression for $\sigma+\sigma'$.

I think that the analysis given in Notes 3 and 4 proves definitely that the differences between the results obtained by Walker on the one hand, and by Lorentz and myself on the other, cannot be attributed to the use of the quasi-stationary assumption or of the energy method by Lorentz, neither of which is used by myself, but are due to the differences in the fundamental assumptions made by us, both as to the constitution of the electron and as to the character of its motion. In view of these results, it seems to me that the two electron models ought to be regarded as essentially distinct, and that the relative merits of the two sets of mass formulæ should be assessed in the light of their agreement with the results of experiment and of their adaptability to the purposes of theory.

On an Expansion of the Point-Potential.

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Introduction.

By point-potential in this paper is meant the potential scalar or vector of a point-charge which is moving in any manner without any restriction except that the speed is less than that of light. Its general form is $f(\tau)/r(1-C^{-1}\partial r/\partial\tau)$, where r is the distance of the point-charge at a time τ from the field-point, C the speed of light, and $f(\tau)$ an arbitrary function of τ, τ being given by the equation $C(t-\tau) = r$. The solution for uniform rectilinear motion of the point-charge was given by Sir J. J. Thomson[*] and Heaviside.[†] The solution for a general motion of the point-charge was given by Liénard[‡] and Wiechert.[§] A method of deducing the result by complex integration was given by the writer,[||] and this method is the basis of the present paper. The subsequent literature is extensive and falls, for the most part, under two heads. First, calculations of electromagnetic inertia include, for uniform motion, many papers by Heaviside, Searle, Morton, Abraham and others. For variable motion a typical paper is Sommerfeld.[¶] A Lagrangian expansion appears to have been given first by Herglotz,[**] by Schott,[††] and

[*] 'Phil. Mag.,' April 1881.

[†] 'Phil. Mag.,' April, 1889.

[‡] 'L'Éclairage Électrique,' 1898.

[§] 'Archives Néerlandaises,'1900, p. 549.

[||] 'Proc. London Math. Soc.,' Series 2, vol. 1.

[¶] 'Gött. Nach.,' 1904, pp. 107–110.

[**] 'Gött. Nach.,' 1903, p. 356.

[††] 'Ann. d. Physik,' vol. 25, p. 63 (1908).